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EQUIVALENCE OF TODA–HOPF INVARIANTS†

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ABSTRACT
In this paper it is shown that various existing constructions of 'Hopf invariant' are equivalent to each other. In consequence one gets that the one *Toda–Hopf invariant* enjoys various properties.

0. Introduction

In 1956, Toda [7] introduced his Hopf invariant. Since then other authors have given alternate constructions of this invariant. In this paper we show that these constructions are equivalent. Each construction has certain properties which are convenient. Hence, it is a consequence of this paper that there is one Toda–Hopf invariant with all of these properties.

First, there is the original definition of Toda which was expressed in modern form by Selick in [6]. Let p be an odd prime. If $J_{p-1}(S^{2n})$ denotes the $(p-1)$-st filtration of the James construction, then the Toda–Hopf invariant is a map $\Omega J_{p-1}(S^{2n-1}) \to \Omega S^{2np-1}$ for which the homotopy theoretic fibre localized at p is S^{2n-1}. This definition is based on a retraction of $\Omega(J_{p-1}(S^{2n}) \vee S^{2n(p-1)})$ onto the homotopy theoretic fibre of the natural map

$$\Omega(J_{p-1}(S^{2n}) \vee S^{2n(p-1)}) \to \Omega(J_{p-1}(S^{2n}) \times S^{2n(p-1)}).$$

As Toda realized, this definition is natural with respect to self maps of S^{2n} and this fact leads to certain consequences concerning the exponents of the odd primary components of the homotopy groups of spheres.

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Second, Gray [2] gave a definition which he based on a retraction of $\Omega J_{p-1}(S^{2n})$ onto the homotopy theoretic fibre of the natural map $\Omega J_{p-1}(S^{2n}) \to \Omega S^{2n(p-1)}$. He has shown that this definition gives a Toda–Hopf invariant which when localized at p is a H-map. We show in Section 3 that this definition may be based on a retraction of $\Omega(J_{p-1}(S^{2n}) \vee S^{2n(p-1)})$ onto the homotopy fibre of the natural map $\Omega(J_{p-1}(S^{2n}) \vee S^{2n(p-1)}) \to \Omega S^{2n(p-1)}$.

Third, Gray [4] gave a construction of a classifying space for the fibre of the double suspension $\Sigma^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$. Using this, he gave another construction of a Toda–Hopf invariant. We show in Section 4 that his construction may be based on a retraction of $\Omega(J_{p-1}(S^{2n}) \vee S^{2n(p-1)})$ onto the homotopy theoretic fibre of the natural map

$$\Omega(J_{p-1}(S^{2n}) \vee S^{2n(p-1)}) \to \Omega(J_{p-1}S^{2n}).$$

In the course of this, we give an alternate but equivalent construction of a classifying space for the fibre of the double suspension.

In Section 2, we show that equivalent Toda–Hopf invariants come about from constructions based on retractions of $\Omega(J_{p-1}(S^{2n}) \vee S^{2n(p-1)})$ onto the three homotopy theoretic fibres mentioned in the three preceding paragraphs. Section 1 contains preliminary facts concerning these fibres and certain evaluation maps.

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1. Evaluation maps and certain fibrations

In this paper space will mean well pointed space having a compactly generated topology.† Connected will mean pathwise connected. The 1-sphere will be denoted S^1 and it will be supposed to be the unit interval with end points collapsed, i.e. $S^1 = I/\{0, 1\}$. If X and Y are spaces, the space of maps from X to Y will be denoted by $\mathrm{Map}(X, Y)$. The reduced suspension of X is the space $S^1 \wedge X$, and is denoted ΣX. The path space of X is the subspace $P(X)$ of $\mathrm{Map}(I, X)$ consisting of those paths starting at the base point $*$. The canonical map $\pi : P(X) \to X$ such that $\pi(\varphi) = \varphi(1)$ is a fibre map with fibre $\Omega(X) = \pi^{-1}(*)$, the space of loops in X. The pair $(P(X), \Omega(X))$ is an NDR pair.

In studying Toda and related Hopf invariants one looks often at the

† Usually the convention and often the notation of [7] will be followed.

following situation: suppose that A and B are connected spaces. There is a diagram

$$\begin{array}{ccc} & A \vee B & \\ & \downarrow \iota & \\ A \xleftarrow{p} & A \times B & \xrightarrow{q} B \end{array}$$

where ι is the canonical inclusion, and p and q are the projections to the factors. Then one makes ι, $p\iota$, and $q\iota$ into fibrations and looks at the induced maps between the fibres, and at combinations with suspensions, loops, evaluations, etc. To this end it will be convenient to look at some special joins and half smashes.

Special joins and half smashes of spaces X_1 and X_2 are defined when, instead of just spaces, one has NDR pairs (C_1, X_1) and (C_2, X_2) such that C_1 and C_2 are contractible as pointed spaces. In this situation the special join of X_1 and X_2 is the space $C_1 \times X_2 \cup X_1 \times C_2$. Abusing notation it is denoted by $X_1 \bar{*} X_2$, except in the standard case where C_1 is the reduced cone over X_1 and C_2 is the reduced cone over X_2 and one obtains $X_1 * X_2$, the reduced join. The left half smash of X_1 and X_2 is

$$X_1 \ltimes X_2 = X_1 \times X_2 \backslash X_1 \times *,$$

the lifted left half smash is

$$X_1 \bar{\ltimes} X_2 = X_1 \times X_2 \cup C_1 \times *,$$

the right half smash is

$$X_1 \rtimes X_2 = X_1 \times X_2 \backslash * \times X_2,$$

and the lifted right half smash is

$$X_1 \bar{\rtimes} X_2 = X_1 \times X_2 \cup * \times C_2.$$

The canonical maps $X_1 \bar{\ltimes} X_2 \to X_1 \ltimes X_2$ and $X_1 \bar{\rtimes} X_2 \to X_1 \rtimes X_2$ are homotopy equivalences, since each is obtained by taking an NDR pair with contractible subspace and smashing the subspace to a point.

The case of the preceding of particular interest is that where A and B are connected spaces, and the pairs in question are $(P(A), \Omega(A))$ and $(P(B), \Omega(B))$. In this case the space $\Omega(A) \bar{*} \Omega(B)$ is called the path join of $\Omega(A)$ and $\Omega(B)$.

If $f: X \to Y$ is a map with Y connected, we recall the standard way of making f into a fibration. Let

$$E_f = \{(x, \varphi) \mid x \in X, \varphi : I \to Y, \text{ and } f(x) = \varphi(1)\},$$

$\pi_f : E_f \to Y$ is defined by $\pi_f(x, \varphi) = \varphi(0)$, and $\iota_f : X \to E_f$ is defined by $\iota_f(x) = (x, e_{f(x)})$ where $e_{f(x)}$ is the constant path at $f(x)$. The space X is identified using ι_f with its image in E_f. Note before making this identification that E_f is topologized as a subspace of $X \times \mathrm{Map}(I, Y)$. Now X is a deformation retract of E_f, $\pi_f \iota_f = f$, and π_f is a fibration with fibre

$$F_f = \{(x, \varphi) \mid (x, \varphi) \in E_f \text{ and } \varphi(0) = *\}.$$

Thus there is a pull back diagram

$$\begin{array}{ccc} F_f & \xrightarrow{\tilde{f}} & P(Y) \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

where $\tilde{f}(x, \varphi) = \varphi$, $\pi'(x, \varphi) = x$. This means that F_f is just the total space of the fibration over X induced by f from the path space fibration over Y.

The preceding implies that for A and B connected when the inclusion $\iota : A \vee B \to A \times B$ is made into a fibration, then the path space join $\Omega(A) \,\overline{*}\, \Omega(B)$ is the fibre.

Suppose now $A \vee B \to A$ is made into a fibration. That part of the fibre lying over A is just $P(A)$, while that part over B is $\Omega(A) \times B$ since $B \to A$ is trivial. These match along $\Omega(A)$ sitting over $*$, and one has that the fibre is $\Omega(A) \,\overline{\ltimes}\, B$. Similarly, when $A \vee B \to B$ is made into a fibration the fibre is $A \,\overline{\rtimes}\, \Omega(B)$.

In this situation one has natural maps

$$e_A : \Omega(A) \,\overline{*}\, \Omega(B) \to \Omega(A) \wedge B \quad \text{and} \quad e_B : \Omega(A) \,\overline{*}\, \Omega(B) \to A \wedge \Omega(B).$$

The first of these is the composite of the induced map between fibres $\Omega(A) \,\overline{*}\, \Omega(B) \to \Omega(A) \ltimes B$ followed by the natural map $\Omega(A) \,\overline{\ltimes}\, B \to \Omega(A) \ltimes B \to \Omega(A) \wedge B$, and $e_A(\lambda, \delta) = \lambda \wedge \delta(1)$. Obtaining e_B similarly one has $e_B(\lambda, \delta) = \lambda(1) \wedge \delta$.

Now observe that $(\Omega(A) \,\overline{*}\, \Omega(B), P(A) \vee P(B))$ is an NDR pair with contractible subspace. Pinching the subspace to a point one obtains the special suspension $\Sigma' \Omega(A) \wedge \Omega(B)$, which is the union of contractible subspaces $P(A) \wedge \Omega(B)$ and $\Omega(A) \wedge P(B)$ meeting in $\Omega(A) \wedge \Omega(B)$ and where all appropriate pairs are contractible.

For φ a path and $s \in I$, let φ_s be the path such that $\varphi_s(t) = \varphi(st)$. Next define

$$\alpha : \Sigma \Omega(A) \wedge \Omega(B) \to \Sigma' \Omega(A) \wedge \Omega(B)$$

by

$$\alpha([t]\wedge\lambda\wedge\delta)=\begin{cases}\lambda_{2t}\wedge\delta & \text{for } 0\leqq t\leqq \frac{1}{2},\\ \lambda\wedge\delta_{2-2t} & \text{for } \frac{1}{2}\leqq t\leqq 1.\end{cases}$$

Observe that α is a homotopy equivalence.

The maps e_A and e_B pass to quotients and composing with α, and again abusing notation, one has

$$e_A:\Sigma\Omega(A)\wedge\Omega(B)\to\Omega(A)\wedge B,$$

$$e_B:\Sigma\Omega(A)\wedge\Omega(B)\to A\wedge\Omega(B)$$

where $e_A([t]\wedge\lambda\wedge\delta)$ is $*$ for $0\leqq t\leqq\frac{1}{2}$ and $\lambda\wedge\delta(2-2t)$ for $\frac{1}{2}\leqq t\leqq 1$, and $e_B([t]\wedge\lambda\wedge\delta)=\lambda(2t)\wedge\delta$ for $0\leqq t\leqq\frac{1}{2}$ and $*$ for $\frac{1}{2}\leqq t\leqq 1$.

Now define

$$e'_A:\Sigma\Omega(A)\wedge\Omega(B)\to\Omega(A)\wedge B \quad\text{by}\quad e'_A([t]\wedge\lambda\wedge\delta)=\lambda\wedge\delta(1-t),$$

and

$$e'_B:\Sigma\Omega(A)\wedge\Omega(B)\to A\wedge\Omega(B) \quad\text{by}\quad e'_B([t]\wedge\lambda\wedge\delta)=\lambda(t)\wedge\delta.$$

Normalization Lemma. *With the definitions above, e_A and e'_A are homotopic, and e_B and e'_B are homotopic.*

Define $D: I\times\Sigma\Omega(A)\wedge\Omega(B)\to\Omega(A)\wedge B$ by

$$D(s,[t]\wedge\lambda\wedge\delta)=\begin{cases}\lambda\wedge\delta(1-st), & 0\leqq t\leqq\frac{1}{2}\\ \lambda\wedge\delta(2-2t-s+st), & \frac{1}{2}\leqq t\leqq 1.\end{cases}$$

Observe that D is a homotopy from e_A to e'_A. A similar homotopy connects e_B to e'_B.

Next, for X a space recall the standard evaluation map $\mathrm{eval}:\Sigma\Omega(X)\to X$ where $\mathrm{eval}([s]\wedge\lambda)=\lambda(s)$, and its homotopy inverse $-\mathrm{eval}:\Sigma\Omega(X)\to X$ where $-\mathrm{eval}([s]\wedge\lambda)=\lambda(1-s)$.

Returning to the situation where A and B are connected spaces, $\mathrm{eval}\wedge 1_B:\Sigma\Omega(A)\wedge B\to A\wedge B$, and $1_A\wedge\mathrm{eval}: A\wedge\Sigma\Omega(B)\to A\wedge B$; combining these with $\Sigma e'_A$ and $\Sigma e'_B$ one obtains

$$\theta_A,\theta_B:\Sigma^2\Omega(A)\wedge\Omega(B)\to A\wedge B,$$

where $\theta_A=(\mathrm{eval}\wedge 1_B)\circ\Sigma e'_A$ and $\theta_B=(1_A\wedge\mathrm{eval})\circ\Sigma e'_B$. Then

$$\theta_A([s]\wedge[t]\wedge\lambda\wedge\delta)=\lambda(s)\wedge\delta(1-t),\quad\text{and}\quad \theta_B([s]\wedge[t]\wedge\lambda\wedge\delta)=\lambda(t)\wedge\delta(s).$$

PROPOSITION. *If A and B are connected spaces*

$$\theta_A, \theta_B : \Sigma^2\Omega(A) \wedge \Omega(B) \to A \wedge B \quad \textit{are homotopic.}$$

First note that $\beta : S^2 = S^1 \wedge S^1 \to S^2$ defined by $\beta([s] \wedge [t]) = [t] \wedge [1 - s]$ is of degree 1, and hence homotopic to the identity. Since $\theta_B = \theta_A \circ \beta$ the proposition follows.

2. Splittings, Toda–Hopf invariants, Gray invariants, and relations between these

Suppose that A and B are simply connected so that $\Omega(A)$ and $\Omega(B)$ are connected. Assume $\pi : E \to A \times B$ is a fibration with simply connected fibre F, $f_A : A \to E$, $\pi f_A(a) = (a, *)$ for $a \in A$, $f_B : B \to E$, $\pi f_B(b) = (*, b)$ for $b \in B$, and $j : F \to E$ is the inclusion. Now one has that the composite

$$\Omega(A) \times \Omega(F) \times \Omega(B) \xrightarrow{\alpha} \Omega(E) \times \Omega(E) \times \Omega(E) \xrightarrow{\varphi_3} \Omega(E)$$

is at least a weak homotopy equivalence where $\alpha = \Omega(f_A) \times \Omega(j) \times \Omega(f_B)$ and φ_3 is the threefold multiplication map.

Either assuming the homotopy type of CW-complexes or inverting weak homotopy equivalences to form the homotopy category one has a natural map $\Omega(E) \to \Omega(F)$ obtained by taking the inverse of the above and projecting on the factor $\Omega(F)$. Next observe that $p\pi : E \to A$ is a fibration with fibre $F_A = \pi^{-1}(* \times B)$, and that f_A is a section of $p\pi$. Letting $j_A : F_A \to E$ be the inclusion one has that the composite

$$\Omega(A) \times \Omega(F_A) \xrightarrow{\beta} \Omega(E) \times \Omega(E) \xrightarrow{\varphi_2} \Omega(E)$$

is an isomorphism in the homotopy category where $\beta = \Omega(f_A) \times \Omega(j_A)$. However, one also has that the composite

$$\Omega(F) \times \Omega(B) \xrightarrow{\delta} \Omega(F_A) \times \Omega(F_A) \xrightarrow{\theta_2} \Omega(F_A)$$

is an isomorphism where $\delta = \Omega(j) \times \Omega(fj)$. The composite $\Omega(E) \to \Omega(F_A) \to \Omega(F)$ is the map obtained earlier.

Proceeding similarly with $q\pi : E \to B$, and $\pi^{-1}(A \times *) = F_B$, one has the homotopy equivalence $\Omega(F_B) \times \Omega(B) \to \Omega(E)$, and $\Omega(E) \to \Omega(F_B)$ in the homotopy category. Then the homotopy equivalence $\Omega(A) \times \Omega(F) \to \Omega(F_B)$, and the composite $\Omega(E) \to \Omega(F_B) \to \Omega(F)$ is the same map as before in the homotopy category.

In the preceding section the diagram

$$\begin{array}{ccccc} & & A \vee B & & \\ & & \downarrow{\scriptstyle i} & & \\ A & \xleftarrow{p} & A \times B & \xrightarrow{q} & B \end{array}$$

was considered, and the maps $\imath$, pi, and qi were made into fibrations. The considerations of the preceding paragraph apply to $\pi_i : E_i \rightarrow A \times B$. The fibre of π_i is canonically equivalent with $\Sigma\Omega(A) \wedge \Omega(B)$, and since the homotopy type of $A \vee B$ is that of E_i canonically one has the pre-Toda–Hopf invariant $T' : \Omega(A \vee B) \rightarrow \Omega(\Sigma\Omega(A) \wedge \Omega(B))$. Further, one had that the fibre of pi was equivalent with $\Omega(A) \ltimes B$; the considerations of the preceding paragraph show that $\Omega(\Omega(A) \ltimes B)$ is equivalent with $\Omega(\Sigma\Omega(A) \wedge \Omega(B)) \times \Omega(B)$. Similarly, looking at qi one has $\Omega(A \rtimes \Omega(B))$ is equivalent with $\Omega(A) \times \Omega(\Sigma\Omega(A) \wedge \Omega(B))$.

In order to obtain Toda–Hopf invariants the space A must have more structure. However, before proceeding the Gray invariants [3] will be introduced.

Thus suppose that (X, Y) is an NDR pair with X and Y connected. Let A be the union of X and the cone over Y. There is a natural map $\theta : A \rightarrow \Sigma Y$ which smashes X to a point. Gray's procedure is to take a combinatorial model for the homotopy fibre of θ, and then to construct various maps with this as domain. Here a direct study of the homotopy fibre will be substituted. Hence let

$$\begin{array}{ccc} F & \xrightarrow{\bar{\theta}} & P(\Sigma Y) \\ \downarrow{\scriptstyle \pi_\theta} & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\theta} & \Sigma Y \end{array}$$

be a pull back diagram. Thus F is a fibration over A with fibre $\Omega(\Sigma Y)$, and that part of the fibration which lies over X is trivial since $\theta(X) = *$. Thus there is a canonical inclusion $\imath : X \times \Omega(\Sigma Y) \rightarrow F$. Since $\Sigma(X \times \Omega(\Sigma Y))$ is canonically homotopy equivalent with $\Sigma X \vee \Sigma\Omega(\Sigma Y) \vee \Sigma X \wedge \Omega(\Sigma Y)$, there is induced a map $\imath' : \Sigma X \vee \Sigma X \wedge \Omega(\Sigma Y) \rightarrow \Sigma F$.

PROPOSITION. *The map $\imath' : \Sigma X \vee \Sigma X \wedge \Omega(\Sigma Y) \rightarrow \Sigma F$ is a weak homotopy equivalence which is a homotopy equivalence if (X, Y) has the homotopy type of a CW pair.*

PROOF. Suppose homology means homology with coefficients in a field. Filter $P(\Sigma Y)$ by letting filtration zero be $\Omega(\Sigma Y) = \pi^{-1}(*)$, and filtration one be

$P(\Sigma Y)$, and filter F by taking inverse images under $\tilde{\theta}$. These result in two small homology spectral sequences. In that for $P(\Sigma Y)$,

$$E^1_{0,*} = H_*(\Omega(\Sigma Y)), \qquad E^1_{1,*} = H_*(\Sigma Y, *) \otimes H_*(\Omega(\Sigma Y)),$$

and $d^1 : E^1_{1,*} \xrightarrow{\approx} \dot{H}_*(\Omega(\Sigma Y)) \subset E^1_{0,*}$ where a dot over homology indicates reduced homology. In that for F,

$$E^1_{0,*} = H_*(X) \otimes H_*(\Omega(\Sigma Y)) = H_*(\Omega(\Sigma Y)) \oplus \dot{H}_*(X) \otimes H_*(\Omega(\Sigma Y)),$$

$E^1_{1,*} = H_*(A, X) \otimes H_*(\Omega(\Sigma Y))$, and essentially by excision $H_*(A, X) \xrightarrow{\approx} H_*(\Sigma Y, *)$. Thus $d^1 : E^1_{1,*} \xrightarrow{\approx} \dot{H}_*(\Omega(\Sigma Y)) \subset E^1_{0,*}$, and $\dot{H}_*(F)$ is isomorphic with $\dot{H}_*(X) \otimes H_*(\Omega(\Sigma Y))$. This means that ι' induces a homology isomorphism with coefficients in any field and thus integrally. Since both the range and domain of ι' are simply connected the proposition follows readily.

The map ι' will be assumed inverted in an appropriate homotopy category.

COROLLARY. *The map $\bar{\iota} : \Sigma X \wedge \Omega(\Sigma Y) \to \Sigma(F/X)$ induced by ι' is an equivalence.*

In the preceding situation the *Gray invariant* is the map $G : F \to \Omega(X \wedge \Sigma Y)$ obtained by taking the natural map $F \to \Omega(\Sigma F)$ and following it by $\alpha : \Omega(\Sigma F) \to \Omega(X \wedge \Sigma Y)$, where α is obtained by taking the composite

$$\Sigma F \to \Sigma(F/X) \to \Sigma X \wedge \Omega(\Sigma Y) \xrightarrow{\text{eval}} X \wedge \Sigma Y$$

and looping it.

Higher Gray invariants [2], [3] may be defined by using an equivalence of ΣF with the infinite bouquet $\bigvee_{n \geq 0} \Sigma(X \wedge \Lambda^n Y)$ rather than by using his combinatorial equivalent of F. These are compatible with James–Hopf invariants obtained by using a corresponding equivalence of $\Sigma\Omega(\Sigma Y)$ with $\bigvee_{n \geq 0} \Sigma\Lambda^n Y$, e.g. [6].

In this situation there is a coaction map $\psi : A \to A \vee \Sigma Y$. Its definition is now recalled.

The cone over Y, $C(Y)$ is $I \times Y/\{1\} \times Y \cup I \times *$, and the map $\psi : C(Y) \to C(Y) \vee \Sigma Y$ is defined by

$$\psi[t, y] = \begin{cases} [2t, y] \in C(Y) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ [2t-1] \wedge y \in \Sigma Y & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Now there are push out diagrams

$$\begin{array}{ccc} Y & \longrightarrow & C(Y) \\ \downarrow & & \downarrow \\ X & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & C(Y)\vee\Sigma Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & A\vee\Sigma Y \end{array}$$

and a map from the first to the second which is the identity on the left column and ψ at the upper right corner. The coaction $\psi : A \to A \vee \Sigma Y$ is the resulting map of the lower right corners. Using also the comultiplication in ΣY one obtains a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A\vee\Sigma Y \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta\vee\Sigma Y} \\ \Sigma Y & \xrightarrow{\psi} & \Sigma Y\vee\Sigma Y \end{array}$$

Let $h : \Sigma Y \to \Sigma Y$ be the composite $\Sigma Y \xrightarrow{\psi} \Sigma Y \vee \Sigma Y \xrightarrow{*\vee\Sigma Y} \Sigma Y$, and let $\theta' : A \to \Sigma Y$ be $h\theta$.

Recalling the procedure for making maps into fibrations from the first paragraph one has a commutative diagram

$$\begin{array}{ccc} F = F_\theta & \longrightarrow & F_{\theta'} \\ \downarrow & & \downarrow \\ E_\theta & \xrightarrow{\tilde h} & E_{\theta'} \\ \downarrow{\scriptstyle \pi_\theta} & & \downarrow{\scriptstyle \pi_{\theta'}} \\ \Sigma Y & \xrightarrow{h} & \Sigma Y \end{array}$$

where the columns are fibrations and $\tilde h(a, \varphi) = (a, h \circ \varphi)$. Since the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A\vee\Sigma Y \\ \downarrow{\scriptstyle \theta'} & & \downarrow{\scriptstyle *\perp\Sigma Y - qi} \\ \Sigma Y & \xrightarrow{\Sigma Y} & \Sigma Y \end{array}$$

commutes, one has a commutative diagram

$$\begin{array}{ccc} F_{\theta'} & \longrightarrow & F_{qi} \\ \downarrow & & \downarrow \\ E_{\theta'} & \longrightarrow & E_{qi} \\ \downarrow{\scriptstyle \pi_{\theta'}} & & \downarrow{\scriptstyle \pi_{qi}} \\ \Sigma Y & \xrightarrow{\Sigma Y} & \Sigma Y \end{array}$$

where the columns are fibrations. Recall that F_{qi} is naturally equivalent with

$A \rtimes \Omega(\Sigma Y)$ which maps naturally to $A \wedge \Omega(\Sigma Y)$. Since π_{qi} has a natural cross section, $\Omega(E_{qi})$ splits as earlier and one obtains

$$\Omega(A) \to \Omega(E_\theta) \to \Omega(E_{\theta'}) \to \Omega(E_{qi}) \to \Omega(F_{qi}) \to \Omega(A \wedge \Omega(\Sigma Y)).$$

However $\Sigma A \wedge \Omega(\Sigma Y) \to A \wedge \Sigma Y$ by evaluation, and so $A \wedge \Omega(\Sigma Y) \to \Omega(\Sigma A \wedge \Omega(\Sigma Y)) \to \Omega(A \wedge \Sigma Y)$. Looping this and comparing with the preceding gives $T_1 : \Omega(A) \to \Omega^2(A \wedge \Sigma Y)$ which is the Gray version of a Toda–Hopf invariant. An elementary argument shows that the diagram

$$\begin{array}{ccc} \Omega(F) & \xrightarrow{\Omega(G)} & \Omega^2(X \wedge \Sigma Y) \\ \downarrow & & \downarrow \\ \Omega(A) & \xrightarrow{T_1} & \Omega^2(A \wedge \Sigma Y) \end{array}$$

commutes in the homotopy category.

The fibration π_{qi} is equivalent with that $q\pi_i$, and in the homotopy category T_1 can be obtained in the fashion of Toda. One has

$$\Omega(A) \xrightarrow{\Omega(\psi)} \Omega(A \vee \Sigma Y) \xrightarrow{T'} \Omega(\Sigma\Omega(A) \wedge \Omega(\Sigma Y))$$

where T' is the pre-Toda–Hopf invariant. Further $\Sigma\Omega(A) \wedge \Omega(\Sigma Y) \to A \wedge \Omega(\Sigma Y)$, and so using one of the compositions above, $\Sigma\Omega(A) \wedge \Omega(\Sigma Y) \to \Omega(A \wedge \Sigma Y)$. Looping this and composing one obtains T_1 in the homotopy category.

In the situation at hand there is another and possibly more basic Toda–Hopf invariant $T_2 : \Omega(A) \to \Omega(A \wedge Y)$. It is obtained by composing

$$\Omega(\Sigma\Omega(A) \wedge \Omega(\Sigma Y)) \xrightarrow{\Omega(\text{eval})} \Omega(\Omega(A) \wedge \Sigma Y) \xrightarrow{\Omega(\text{eval})} \Omega(A \wedge Y)$$

with $T' \circ \Omega(\psi)$. Now $\iota : \Omega(A \wedge Y) \to \Omega^2(\Sigma A \wedge Y) \to \Omega^2(A \wedge \Sigma Y)$ and composing with T_2 gives the negative of T_1 in the homotopy category. Indeed, expressing one of the pieces of T_1 in an adjoint form, one has $\Sigma\Omega(A) \wedge \Sigma Y \to A \wedge \Sigma Y$ is given by $[r] \wedge \varphi \wedge [s] \wedge y \to \varphi(r) \wedge [s] \wedge y$ and a piece of ιT_2 is given by $[r] \wedge \varphi \wedge [s] \wedge y \to \varphi(s) \wedge [r] \wedge y$. Thus one may be obtained from the other by composing with an interchange of suspension coordinates which is of degree -1 on an appropriate two sphere.

3. The James construction and James and Toda–Hopf invariants

First recall some standard properties of the James construction $J(X)$ of a space X ([4], [5]). As a set it is the free monoid generated by X with the relation $* = 1$, as a space it is the colimit of the subspaces $J_n(X)$ where $J_n(X)$ consists of the words of length $\leqq n$, topologized as a quotient of the n-fold product of X with itself. For $n = 0$, $J(X) = *$, and for $n > 0$, $J_n(X)/J_{n-1}(X) = \Lambda^n(X)$. For X connected there is up to homotopy a well-defined map $J(X) \to \Omega(\Sigma X)$ which is an equivalence of group like objects in the homotopy category. If $X = \Sigma Y$, and $n \geqq 1$, it may be assumed that $J_n(\Sigma Y)$ is obtained from $J_{n-1}(\Sigma Y)$ via a pushout diagram

$$\begin{array}{ccc} \Sigma^{n-1}\Lambda^n Y & \to & C(\Sigma^{n-1}\Lambda^n Y) \\ \downarrow & & \downarrow \\ J_{n-1}(\Sigma Y) & \to & J_n(\Sigma Y) \end{array}$$

Thus there is a coaction map $J_n(\Sigma Y) \to J_n(\Sigma Y) \vee \Sigma^n \Lambda^n Y$. The considerations of the preceding sections apply. This means that there is a Gray–Toda–Hopf invariant

$$T_1 : \Omega(J_n(\Sigma Y)) \to \Omega^2(J_n(\Sigma Y) \wedge \Sigma^n \Lambda^n Y),$$

and a Toda–Hopf invariant

$$T_2 : \Omega(J_n(\Sigma Y)) \to \Omega(J_n(\Sigma Y) \wedge \Sigma^{n-1} \Lambda^n Y).$$

However, $J_n(\Sigma Y) \wedge \Sigma^n \Lambda^n Y \to \Omega(\Sigma^2 Y) \wedge \Sigma^n \Lambda^n Y$, and $\Omega(\Sigma^2 Y) \wedge \Sigma^n \Lambda^n Y \to \Sigma\Omega(\Sigma^2 Y) \wedge \Sigma^{n-1}\Lambda^n Y \xrightarrow{\text{(eval)}} \Sigma^{n+1}\Lambda^{n+1} Y$. Thus there is an extended Gray–Toda–Hopf invariant $\bar{T}_1 : \Omega(J_n(\Sigma Y)) \to \Omega^2(\Sigma^{n+1}\Lambda^{n+1} Y)$ which is the double loop of the preceding triple composite $J_n(\Sigma Y) \wedge \Sigma^n \Lambda^n Y \to \Sigma^{n+1}\Lambda^{n+1} Y$ composed with T_1.

If $n \geqq 2$ there is also the composition

$$J_n(\Sigma Y) \wedge \Sigma^{n-1}\Lambda^n Y \to \Omega(\Sigma^2 Y) \wedge \Sigma^{n-1}\Lambda^n Y \to \Sigma^n \Lambda^{n+1} Y.$$

Looping and composing with T_2 gives rise to the extended Toda–Hopf invariant $\bar{T}_2 : \Omega(J_n(\Sigma Y)) \to \Omega(\Sigma^n \Lambda^{n+1} Y)$. The composite of $\Omega(\Sigma^n \Lambda^{n+1} Y) \to \Omega^2(\Sigma^{n+1}\Lambda^{n+1} Y)$ with $\bar{T}_2$ gives $\bar{T}_1$, since an additional interchange of factors has occurred.

Until now the most interesting results have been in the case $\Sigma Y = S^{2m}$, $m > 0$. Suppose also that the prime 2 has been inverted. Then $\Omega(S^{2q})$ is canonically equivalent with $S^{2q-1} \times \Omega(S^{4q-1})$ for any $q > 0$. The composite of $\Omega^2(S^{2m(n+1)}) \to \Omega(S^{2m(n+1)-1})$ with $\bar{T}_1$ may be identified with $\bar{T}_2$, and one obtains $T : \Omega(J_n(S^{2m})) \to \Omega(S^{2m(n+1)-1})$. Additional information about T may be easily found (e.g. [2] or [4]). The next question to be considered is which further standard localizations have the property that the homotopy fibre of T is S^{2m-1}, and when does the fibration split. Note both of these things always happen rationally. Start by using the Serre spectral sequence for $P(J_n(S^{2m})) \to J_n(S^{2m})$ to compute the homology of $\Omega(J_n(S^{2m}))$. Now $H_q(J_n(S^{2m})) = 0$ except for $q = 2jm, j = 0, \cdots, n$, and one may assume a basis in degree $2jm$ is x^j as denoted by its image in $H_*(J(S^{2m}))$. In the spectral sequence $d^r = 0$ for $2 \leqq r < 2m$, and $d^{2m}x = a$ a generator of $H_{2m-1}(\Omega(J_n(S^{2m})))$. Thus $d^{2m}x^j = jax^{j-1}$ for $j = 1, \cdots, n$ leading to the conclusion that one wants to invert all primes $\leqq n$ as well as 2, deal with simply connected spaces so localized, and compute with coefficients the integers also so localized. One also assumes $m \geqq 2$. One obtains readily that

$$H_*(\Omega(J_n(S^{2m}))) = E(a, 2m - 1) \otimes S(b, 2m(n + 1) - 2),$$

the tensor product of the exterior algebra with one generator a of degree $2m - 1$, and the polynomial algebra with one generator b of degree $2m(n + 1) - 2$. The only other nontrivial differential in the spectral sequence is d^{2mn} and one may assume that $d^{2mn}(ax^n) = b$ modulo the right ideal generated by a.

Looking at $J_n(S^{2m}) \to S^{2mn}$ one may suppose in homology $x^n \to \bar{x}^n$ a generator of $H_{2mn}(S^{2mn})$. A standard calculation shows $H_*(\Omega(S^{2mn})) = T(c, 2mn - 1)$, the tensor algebra on one generator c of degree $2mn - 1$. As a Hopf algebra $H_*(\Omega(J_n(S^{2m}) \vee S^{2mn}))$ is the coproduct of the sub-Hopf algebras $H_*(\Omega(J_n(S^{2m})))$ and $H_*(\Omega(S^{2mn}))$. A basis for the primitive elements in degree $2m(n + 1) - 2$ is b and $[a, c] = ac + ca$. The coaction map induces a map of spectral sequences of path fibrations, $d^{2mn}\bar{x}^n = c$, and $ax^n \to ax^n + a\bar{x}^n$ which goes into $b + ac$ modulo the right ideal generated by a. Since b is primitive, this means $H_*(\Omega(\psi))(b) = b + [a, c]$. Thus if $H_*(\Omega(S^{2m(n+1)-1})) = S(b_1, 2m(n + 1) - 2)$, it may be assumed $H_*(\bar{T})(b) = b_1$, which implies that the homotopy fibre of $\bar{T}$ is S^{2m-1}. Suppose $(n + 1)$ is not a prime; there is a map $S^{2m(n+1)-1} \to J_n(S^{2m})$ which may be used to attach a cell to obtain $J_{n+1}(S^{2m})$. Looping and composing

$$\Omega(S^{2m(n+1)-1}) \to \Omega(J_n(S^{2m})) \xrightarrow{\ T\ } \Omega(S^{2m(n+1)-1}),$$

and the composite must be an equivalence in order to obtain the appropriate homology for $\Omega(J_{n+1}(S^{2m}))$.

Proposition. *Assuming primes less than or equal to n and 2 have been invented, then for $m > 1$*

(1) *if* $(n+1)$ *is not a prime* $\Omega(J_n(S^{2m}))$ *is equivalent with* $S^{2m-1} \times \Omega(S^{2m(n+1)-1})$ *in the homotopy category, and*

(2) *if* $(n+1) = p$ *is a prime there is a prefibration sequence* $S^{2m-1} \to \Omega(J_{p-1}(S^{2m})) \xrightarrow{T} \Omega(S^{2pm-1})$.

The proposition follows readily from the preceding discussion. It is a slight strengthening of an old result of Toda [7].

COROLLARY. *For* p *a prime greater than* $(n+1)$, *the exponent of the homotopy of* $J_n(S^{2m})$ *at* p *is* $p^{m(n+1)-1}$ *and the exponent of the homotopy of* $J_{p-1}(S^{2m})$ *at* p *divides* $p^{(p+1)m-2}$.

PROPOSITION (Gray). *Localized at* p, *the Toda–Hopf invariant* $T: \Omega(J_{p-1}(S^{2m})) \to \Omega(S^{2pm-1})$ *for* p *an odd prime is an H-map.*

SKETCH OF PROOF. There is a fibration $F \to E_\theta \to S^{2(p-1)m}$ obtained by making the canonical map $\theta: J_{p-1}(S^{2m}) \to S^{2(p-1)m}$ into a fibration and a resulting Gray invariant $G: F \to \Omega(J_{p-2}(S^{2m}) \wedge S^{2(p-1)m})$. However, one has

$$J_{p-2}(S^{2n}) \wedge S^{2(p-1)m} \to \Omega(S^{2m+1}) \wedge S^{2(p-1)m} \xrightarrow{(\text{eval})} S^{2m+1} \wedge S^{2(p-1)m-1}.$$

Looping this composite and composing with G one has $F \to \Omega(S^{2pm})$, but $\Omega(S^{2pm}) = S^{2pm-1} \times \Omega(S^{4pm-1})$. Thus there is an extended Gray invariant which will still be denoted by $G: F \to S^{2pm-1}$. Now it results immediately from earlier considerations that the diagram

$$\begin{array}{ccc} \Omega(F) & \xrightarrow{\Omega(G)} & \Omega(S^{2pm-1}) \\ \downarrow{\scriptstyle \Omega(\pi)} & \nearrow{\scriptstyle T} & \\ \Omega(J_{p-1}(S^{2m})) & & \end{array}$$

is commutative in the homotopy category. Using work of one of us it can easily be shown that $\Omega(\pi)$ has a section [2, Proposition 7]. Then $\Omega(\pi)$ is an epimorphism in the homotopy category as is $\Omega(\pi) \wedge \Omega(\pi)$. This implies the desired result, i.e., that T is an H-map.

PROPOSITION. *If* p *is an odd prime, and* $m > 1$, *then localized at* p, *there is a map* $h: \Omega^2(S^{2m+1}) \to S^{2m-1}$ *such that the composite*

$$\Omega^2(S^{2m+1}) \xrightarrow{h} S^{2m-1} \xrightarrow{E^2} \Omega^2(S^{2m+1})$$

is multiplication by p^2 *in the homotopy category.*

Sketch of Proof. Recall that up to homotopy there is a fibration sequence

$$J_{p-1}(S^{2m}) \xrightarrow{j} \Omega(S^{2m+1}) \xrightarrow{H} \Omega(S^{2pm+1})$$

where H is a p-th James–Hopf invariant. One has $Hp = p^p H$ ([2], [5]). Thus looking at $\Omega(H)$ which is multiplicative $\Omega(H)p = p^p\Omega(H) = \Omega(H)p^p$, and $\Omega(H)(p - p^p) = 0$. Since $p - p^p = p(1 - p^{p-1})$, and $(1 - p^{p-1})$ is an automorphism in the homotopy category one has $\Omega(H)p = 0$, and there exists $g: \Omega^2(S^{2m+1}) \to \Omega(J_{p-1}(S^{2m}))$ such that $\Omega(j)g = p$. Next, since $p^pT = TJ_{p-1}(p)$, and $p^pT = Tp^p$ since T is an H-map, one has $T(J_{p-1}(p) - p^p) = 0$. Looking at the fibration sequence

$$S^{2m-1} \xrightarrow{i} \Omega(J_{p-1}(S^{2m})) \xrightarrow{T} \Omega(S^{2pm-1}),$$

it follows that there exists $f: \Omega(J_{p-1}(S^{2m})) \to S^{2m-1}$ such that $\imath f = J_{p-1}(p) - p^p$. Recall that $jJ_{p-1}(p) = pj$, and that $\Omega(j)\imath = E^2$. Now one has

$$E^2 fg = \Omega(j)\imath fg = \Omega(j)(J_{p-1}(p) - p^p)g = (p - p^p)\Omega(j)g = p^2(p^{p-1} - 1).$$

Since $p^{p-1} - 1$ is an automorphism, the existence of an h having the desired property follows at once.

A result similar to the above using $\Omega(T)$ and factoring $\Omega(E^2)$ was proved quite some time ago by one of us. However, once one knows that T is an H-map, looping again is not necessary. The preceding proof parallels in some ways the old proof of Toda [7] which gives for homotopy groups the corresponding fact. There is a considerably stronger result to the effect that p factors through the double suspension [1]. However, currently the stronger result seems considerably less elementary.

4. Toda–Hopf invariants and classiyfing spaces for the fibre of the double suspension

In this section all spaces will be assumed localized at an odd prime p. Suppose $2 \leqq k$, $1 \leqq m$, and $j: J_{k-1}(S^{2m}) \to \Omega(S^{2m+1})$, and $\imath: S^{2m-1} \to \Omega(J_{k-1}(S^{2m}))$ the natural maps. Recall from the preceding section that there is a fibration sequence

$$S^{2m-1} \xrightarrow{i} \Omega(J_{k-1}(S^{2n})) \xrightarrow{T} \Omega(S^{2nk-1})$$

for $2 \leqq k \leqq p$ where T is the Toda–Hopf invariant, and that this sequence is split if $k < p$. Recall also that $\Omega(j)i : S^{2m-1} \to \Omega^2(S^{2m+1})$ is the double suspension E^2, and that there is a fibration sequence

$$J_{p-1}(S^{2n}) \xrightarrow{j} \Omega(S^{2m+1}) \xrightarrow{H} \Omega(S^{2mp+1})$$

where H is the p-th James–Hopf invariant. The fibre of the double suspension will be denoted by $C(m)$.

There is a commutative diagram

$$\begin{array}{ccc} J_{k-1}(S^{2m}) \vee S^{2m(k-1)} & \xrightarrow{f'} & J_{k-1}(S^{2m}) \\ \downarrow{\scriptstyle j\vee 1} & & \downarrow{\scriptstyle j} \\ \Omega(S^{2m+1}) \vee S^{2m(k-1)} & \xrightarrow{f} & \Omega(S^{2m+1}) \end{array}$$

where f and f' are the natural maps considered earlier but given a new notation so as not to conflict with that for the fixed prime.

The pre-Toda–Hopf invariant for the bottom line is

$$T' : \Omega(\Omega(S^{2m+1}) \vee S^{2m(k-1)}) \to \Omega(\Sigma\Omega^2(S^{2m+1}) \wedge \Omega(S^{2m(k-1)}))$$

and the range maps naturally to $\Omega(S^{2mk-1})$. Clearly T is just the composite of the latter map with T' and then with $\Omega(j \vee 1)\Omega(\psi)$, where ψ is as before the coaction map.

To proceed further, some facts about shifts of fibration sequences are needed. There are basically two ways to shift a fibration sequence, and they are equivalent. Thus suppose

$$A \xrightarrow{u} B \xrightarrow{\delta} C$$

is a fibration sequence of connected spaces. There is a new fibration sequence

$$F_u \xrightarrow{i_u} E_u \xrightarrow{\pi_u} B$$

obtained by making u into a fibration. This is the shift of the original sequence done by the first procedure. The second procedure will now be recalled. For X a space let $P'(X)$ be the space of paths in X ending at the base point, and $\pi' : P'(X) \to X$ the canonical fibre map defined by $\pi'(\alpha) = \alpha(0)$. Note that this is exactly what would be obtained by making $* \to X$ into a fibration. Returning to the original fibration sequence, let $\theta_\delta : A_\delta \to B$ be the fibration over B induced by δ from the fibration $P'(C) \to C$. Recall that

$$E_u = \{(a, \varphi) \mid a \in A, \varphi : I \to B, \text{ and } u(a) = \varphi(1)\},$$

$\pi_u(a, \varphi) = \varphi(0)$, $A_\delta = \{(b, \eta) \mid b \in B,\ \eta \in P'(C)$, and $\delta(b) = \eta(0)\}$. There is a commutative diagram

$$\begin{array}{ccccc} F_u & \xrightarrow{i_u} & E_u & \xrightarrow{\pi_u} & B \\ \downarrow \delta & & \downarrow \tilde{\delta} & & \downarrow = \\ \Omega(C) & \xrightarrow{j_\delta} & A_\delta & \xrightarrow{\theta_\delta} & B \end{array}$$

where j_δ is the natural inclusion of the fibre, and $\tilde{\delta}(a, \varphi) = (\varphi(0), \delta \circ \varphi)$. The rows are fibration sequences, $\tilde{\delta}$ is a fibration with fibre $P(A)$ which is contractible, and $F_u = \tilde{\delta}^{-1}(\Omega(C))$. The maps δ, $\tilde{\delta}$ are homotopy equivalences, and thus the rows are isomorphic in the homotopy category. The lower row is the shift of the original fibration by the second procedure. Under reasonable hypotheses one may iterate shifting. In particular, if B and C are simply connected the third shift exists and is equivalent with

$$\Omega(A) \xrightarrow{\Omega(u)} \Omega(B) \xrightarrow{\Omega(\delta)} \Omega(C)$$

canonically. Suppose now that one has a section of δ, i.e. $f: C \to B$ such that $\delta f = 1_C$. This splits the fibration sequence of loop spaces, and one has $r: \Omega(B) \to \Omega(A)$ such that $r\Omega(u) = 1_{\Omega(A)}$, $r\Omega(f) = *$, and r is compatible with the left action of $\Omega(A)$ on $\Omega(B)$. In the homotopy category r is the unique morphism with these properties.

The fact that one had a section of δ can be expressed differently. Thus there is a commutative diagram

$$\begin{array}{ccccc} * & \longrightarrow & C & \xrightarrow{=} & C \\ \downarrow & & \downarrow f & & \downarrow = \\ A & \xrightarrow{u} & B & \xrightarrow{\delta} & C \end{array}$$

with the rows fibration sequences. Shifting by the second procedure one obtains a commutative diagram

$$\begin{array}{ccccc} \Omega(C) & \longrightarrow & P'(C) & \longrightarrow & C \\ \downarrow = & & \downarrow & & \downarrow f \\ \Omega(C) & \xrightarrow{j_\delta} & A_\delta & \xrightarrow{\theta_\delta} & B \end{array}$$

Thus j_δ is null homotopic and canonically so. Up to homotopy there is a unique choice of retraction $h: A_\delta \to A$, since there is a canonical map $A \to A_\delta$ which is a homotopy equivalence. Thus there is a commutative diagram

$$\begin{array}{ccc} \Omega(C) & \xrightarrow{j_\delta} & A_\delta \\ \downarrow{\scriptstyle \tilde{j}_\delta} & & \downarrow{\scriptstyle h} \\ P'(A) & \longrightarrow & A \end{array}$$

There results a commutative diagram

$$\begin{array}{ccccc} \Omega(B) & \longrightarrow & \widetilde{\Omega(C)} & \longrightarrow & A_\delta \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \tilde{j}_\delta} & & \downarrow{\scriptstyle h} \\ \Omega(A) & \longrightarrow & P'(A) & \longrightarrow & A \end{array}$$

LEMMA. *The morphism $s : \Omega(B) \to \Omega(A)$ is up to homotopy the canonical retraction $r : \Omega(B) \to \Omega(A)$.*

SKETCH OF PROOF. Abusing notation a little, there is a commutative diagram

$$\begin{array}{ccccc} * & \longrightarrow & \Omega(C) & \xrightarrow{-} & \Omega(C) \\ \downarrow & & \downarrow{\scriptstyle \Omega(f)} & & \downarrow{\scriptstyle -} \\ \Omega(A) & \longrightarrow & \Omega(B) & \longrightarrow & \Omega(C) \\ \downarrow{\scriptstyle -} & & \downarrow{\scriptstyle s} & & \downarrow \\ \Omega(A) & \xrightarrow{-} & \Omega(A) & \longrightarrow & * \end{array}$$

such that the first two rows are the triple shift of the original diagram, and the bottom two rows are the shift of the preceding diagram. This implies the desired result.

Now we return to the considerations at the beginning of the section involving the canonical maps $j : J_{k-1}(S^{2m}) \to \Omega(S^{2m+1})$ and $f : \Omega(S^{2m+1}) \vee S^{2m(k-1)} \to \Omega(S^{2m+1})$. Making both into fibrations one may obtain a commutative diagram

$$\begin{array}{lccccc} (1) & F_{k-1}(m) & \longrightarrow & E_j & \xrightarrow{\pi_j} & \Omega(S^{2m+1}) \\ & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle -} \\ (2) & \Omega^2(S^{2m+1}) \overline{\ltimes} S^{2m(k-1)} & \longrightarrow & E_f & \xrightarrow{\pi_f} & \Omega(S^{2m+1}) \end{array}$$

where the rows are the resulting fibration sequences, the diagram

$$\begin{array}{ccc} J_{k-1}(S^{2m}) & \longrightarrow & E_j \\ \downarrow & & \downarrow{\scriptstyle \beta} \\ \Omega(S^{2m+1}) \vee S^{2m(k-1)} & \longrightarrow & E_f \end{array}$$

is homotopy commutative, and the notation $F_{k-1}(m)$ has been introduced for the fibre of j. The fibration sequence (2) has a canonical cross section. Let $\pi_{k-1}(m): F_{k-1}(m) \to S^{2mk-1}$ be the composite of the canonical map $\Omega^2(S^{2m+1}) \overline{\ltimes} S^{2m(k-1)} \to S^{2mk-1}$ with α, and let $B(m, k)$ be the homotopy fibre of $\pi_{k-1}(m)$. Indeed, this fibration sequence is denoted

$$B(m, k) \to \widetilde{F_{k-1}(m)} \xrightarrow{\widetilde{\pi_{k-1}(m)}} S^{2m(k-1)}.$$

The connecting morphism of the sequence (2) is canonically trivial since it splits canonically. Then the connecting morphism of (1) lifts to $\lambda : \Omega^2(S^{2m+1}) \to B(m, k)$ well defined up to homotopy.

Assuming $k \leqq p$, and abusing notation a little, there results a commutative diagram

$$\begin{array}{ccccc}
S^{2m-1} & \xrightarrow{\rho} & \Omega^2(S^{2m+1}) & \xrightarrow{\lambda} & B(m,k) \\
\downarrow & & \downarrow - & & \downarrow \\
\Omega(J_{k-1}(S^{2m})) & \longrightarrow & \Omega^2(S^{2m+1}) & \longrightarrow & \widetilde{F_{k-1}(m)} \\
\downarrow T & & \downarrow & & \downarrow \widetilde{\pi_{k-1}(m)} \\
\Omega(S^{2mk-1}) & \longrightarrow & P'(S^{2mk-1}) & \longrightarrow & S^{2mk-1}
\end{array}$$

such that the center row is the second shift of (1), the bottom row is the canonical sequence indicated, and the preceding lemma is used with the first part of this section in seeing that the left column is the fibration sequence indicated.

PROPOSITION. *For $2 \leqq k \leqq p$, and $1 \leqq m$, the spaces $B(m, k)$ are classifying spaces for the space $C(m)$ which is the homotopy fibre for the double suspension $E^2 : S^{2m-1} \to \Omega^2(S^{2m+1})$. Moreover, in the homotopy category these spaces are isomorphic with those introduced by Gray.*

SKETCH OF PROOF. Returning to the preceding diagram, the properties already stated imply that the top row is a fibration sequence, and that the connectivity of $B(m, k)$ is greater than $2m$. Then one may assume $\rho = E^2$, so that $B(m, k)$ is as stated. It remains to see that the spaces $B(m, k)$ are equivalent with those of Gray [4]. Thus a version of Gray's construction will be recalled.

Since $J_{k-1}(S^{2m})$ may be obtained from $J_{k-2}(S^{2m})$, by attaching a cell $e^{2m(k-1)}$ by a map $S^{2m(k-1)-1} \to J_{k-2}(S^{2m})$. Thus it may be assumed

$$J_{k-1}(S^{2m}) = J_{k-2}(S^{2m}) \cup e^{2m(k-1)}$$

by use of a mapping cylinder, and that the whole space modulo either subspace gives an NDR pair. Recalling how fibrations are made, there is a commutative diagram

$$\begin{array}{ccccc} \Omega^2(S^{2m+1}) & \longrightarrow & F_{k-1}(m) & \xrightarrow{\pi} & J_{k-1}(S^{2m}) \\ \downarrow = & & \downarrow \alpha & & \downarrow (j \vee 1)\psi \\ \Omega^2(S^{2m+1}) & \longrightarrow & \Omega^2(S^{2m+1}) \overline{\ltimes} S^{2m(k-1)} & \xrightarrow{\pi_1} & \Omega(S^{2m+1}) \vee S^{2m(k-1)} \end{array}$$

such the upper row is essentially the first shift of (1) and the lower row that of (2). Let $F' = \pi^{-1}(J_{k-1}(S^{2m}))$, and $F'' = \pi^{-1}(e^{2m(k-1)})$. Now $F_{k-1}(m)/F' = F''/F' \cap F''$, the pair $(F', F' \cap F'')$ is fibre homotopy equivalent with $\Omega^2(S^{2m-1}) \times (e^{2m(k-1)}, S^{2m(k-1)-1})$, and hence $F''/F' \cap F''$ is homotopy equivalent with $\Omega^2(S^{2m+1}) \overline{\ltimes} S^{2m(k-1)}$. Gray's map $F_{k-1}(m) \to S^{2mk-1}$ is the composite

$$F_{k-1}(m) \to F_{k-1}(m)/F' \to \Omega^2(S^{2m+1}) \overline{\ltimes} S^{2m(k-1)} \to S^{2mk-1}.$$

Since the pair $(J_{k-1}(S^{2m}) \times S^{2m(k-1)}, J_{k-1}(S^{2m}) \vee S^{2m(k-1)})$ is $2mk - 1$ connected one may suppose that after deformation the coaction map $\psi: J_{k-1}(S^{2m}) \to J_{k-1}(S^{2m}) \vee S^{2m(k-1)}$ has the property that if $x \in J_{k-2}(S^{2m})$ then $\psi(x) = (x, *)$. Recall that

$$\Omega^2(S^{2m+1}) \overline{\ltimes} S^{2m(k-1)} = P(\Omega(S^{2m+1})) \cup \Omega^2(S^{2m+1}) \times S^{2m(k-1)},$$

the intersection of the designated subspaces being the fibre $\Omega^2(S^{2m+1})$. Assuming ψ is as above $\alpha(F') \subset P(\Omega(S^{2m+1}))$, α is fibre preserving, and the map $(e^{2m(k-1)}, S^{2m(k-1)-1}) \to (S^{2m(k-1)}, *)$ induced by taking the coaction and then projecting is of degree one. These things combine to mean that α induces a homotopy equivalence $\bar{\alpha}: F''/F' \cap F'' \to \Omega^2(S^{2m+1}) \ltimes S^{2m(k-1)}$ upon smashing $P(\Omega(S^{2m+1}))$ to a point. This shows Gray's map is the same as ours in the homotopy category, and suffices to complete the proof of the proposition.

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